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# A GENERALISATION OF THE TRAPEZOIDAL RULE FOR THE RIEMANN-STIELTJES INTEGRAL AND APPLICATIONS

S.S. DRAGOMIR, C. BUŞE, M.V. BOLDEA, AND L. BRAESCU

ABSTRACT. A generalisation of the trapezoid rule for the Riemann-Stieltjes integral and applications for special means are given.

## 1. INTRODUCTION

The following inequality is well known in the literature as the “*trapezoid inequality*”:

$$(1.1) \quad \left| \frac{f(a) + f(b)}{2} \cdot (b - a) - \int_a^b f(t) dt \right| \leq \frac{1}{12} (b - a)^3 \|f''\|_\infty,$$

where the mapping  $f : [a, b] \rightarrow \mathbb{R}$  is assumed to be twice differentiable on  $(a, b)$ , with its second derivative  $f'' : (a, b) \rightarrow \mathbb{R}$  bounded on  $(a, b)$ , that is,  $\|f''\|_\infty := \sup_{t \in (a, b)} |f''(t)| < \infty$ . The constant  $\frac{1}{12}$  is sharp in (1.1) in the sense that it cannot be replaced by a smaller constant.

If  $I_n : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$  is a division of the interval  $[a, b]$  and  $h_i = x_{i+1} - x_i$ ,  $\nu(h) := \max \{h_i | i = 0, \dots, n-1\}$ , then the following formula, which is called the “*trapezoid quadrature formula*”

$$(1.2) \quad T(f, I_n) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \cdot h_i$$

approximates the integral  $\int_a^b f(t) dt$  with an error of approximation  $R_T(f, I_n)$  which satisfies the estimate

$$(1.3) \quad |R_T(f, I_n)| \leq \frac{1}{12} \|f''\|_\infty \sum_{i=0}^{n-1} h_i^3 \leq \frac{b-a}{12} \|f''\|_\infty [\nu(h)]^2.$$

In (1.3), the constant  $\frac{1}{12}$  is sharp as well.

If the second derivative does not exist or  $f''$  is unbounded on  $(a, b)$ , then we cannot apply (1.3) to obtain a bound for the approximation error. It is important, therefore, that we consider the problem of estimating  $R_T(f, I_n)$  in terms of lower derivatives.

Define the following functional associated to the trapezoid inequality

$$(1.4) \quad \Psi(f; a, b) := \frac{f(a) + f(b)}{2} \cdot (b - a) - \int_a^b f(t) dt$$

where  $f : [a, b] \rightarrow \mathbb{R}$  is an integrable mapping on  $[a, b]$ .

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The following result is known [3]:

**Theorem 1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be an absolutely continuous mapping on  $[a, b]$ . Then*

$$(1.5) \quad |\Psi(f; a, b)| \leq \begin{cases} \frac{(b-a)^2}{4} \|f'\|_\infty & \text{if } f' \in L_\infty[a, b]; \\ \frac{(b-a)^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}} \|f'\|_p & \text{if } f' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{b-a}{2} \|f'\|_1, & \end{cases}$$

where  $\|\cdot\|_p$  are the usual  $p$ -norms, i.e.,

$$\begin{aligned} \|f'\|_\infty &:= \operatorname{ess\,sup}_{t \in [a, b]} |f'(t)|, \\ \|f'\|_p &:= \left( \int_a^b |f'(t)|^p dt \right)^{\frac{1}{p}}, \quad p > 1 \end{aligned}$$

and

$$\|f'\|_1 := \int_a^b |f'(t)| dt,$$

respectively.

The following corollary for composite formulae holds [3].

**Corollary 1.** *Let  $f$  be as in Theorem 1. Then we have the quadrature formula*

$$(1.6) \quad \int_a^b f(x) dx = T(f, I_n) + R_T(f, I_n),$$

where  $T(f, I_n)$  is the trapezoid rule and the remainder  $R_T(f, I_n)$  satisfies the estimation

$$(1.7) \quad |R_T(f, I_n)| \leq \begin{cases} \frac{1}{4} \|f'\|_\infty \sum_{i=0}^{n-1} h_i^2 & \text{if } f' \in L_\infty[a, b]; \\ \frac{1}{2(q+1)^{\frac{1}{q}}} \|f'\|_p \left( \sum_{i=0}^{n-1} h_i^{q+1} \right)^{\frac{1}{q}} & \text{if } f' \in L_p[a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} \|f'\|_1 \nu(h). & \end{cases}$$

A more general result concerning a trapezoid inequality for functions of bounded variation has been proved in the paper [4].

**Theorem 2.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a mapping of bounded variation on  $[a, b]$  and denote  $\bigvee_a^b(f)$  as its total variation on  $[a, b]$ . Then we have the inequality*

$$(1.8) \quad |\Psi(f; a, b)| \leq \frac{1}{2} (b-a) \bigvee_a^b(f).$$

The constant  $\frac{1}{2}$  is sharp in the sense that it cannot be replaced by a smaller constant.

The following corollary which provides an upper bound for the approximation error in the trapezoid quadrature formula, for  $f$  of bounded variation, holds [4].

**Corollary 2.** *Assume that  $f : [a, b] \rightarrow \mathbb{R}$  is of bounded variation on  $[a, b]$ . Then we have the quadrature formula (1.6) where the reminder satisfies the estimate*

$$(1.9) \quad |R_T(f, I_n)| \leq \frac{1}{2} \nu(h) \bigvee_a^b(f).$$

The constant  $\frac{1}{2}$  is sharp.

For other recent results on the trapezoid inequality see [5]-[10], or the book [11] where further references are given.

The following theorem generalizing the classical trapezoid inequality for mappings of bounded variation holds [12].

**Theorem 3.** *Let  $f : [a, b] \rightarrow \mathbb{K}$  ( $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ) be a  $p$ - $H$ -Hölder type mapping, that is, it satisfies the condition*

$$(1.10) \quad |f(x) - f(y)| \leq H |x - y|^p \text{ for all } x, y \in [a, b],$$

where  $H > 0$  and  $p \in (0, 1]$  are given, and  $u : [a, b] \rightarrow \mathbb{K}$  is a mapping of bounded variation on  $[a, b]$ . Then we have the inequality:

$$(1.11) \quad |\Psi(f, u; a, b)| \leq \frac{1}{2^p} H (b - a)^p \bigvee_a^b(u),$$

where  $\Psi(f, u; a, b)$  is the generalized trapezoid functional

$$(1.12) \quad \Psi(f, u; a, b) := \frac{f(a) + f(b)}{2} [u(b) - u(a)] - \int_a^b f(t) du(t).$$

The constant  $C = 1$  on the right hand side of (1.11) cannot be replaced by a smaller constant.

The following corollaries are natural consequences of (1.11):

**Corollary 3.** *Let  $f$  be as above and  $u : [a, b] \rightarrow \mathbb{R}$  be a monotonic mapping on  $[a, b]$ . Then we have*

$$(1.13) \quad |\Psi(f, u; a, b)| \leq \frac{1}{2^p} H (b - a)^p |u(b) - u(a)|.$$

**Corollary 4.** *Let  $f$  be as above and  $u : [a, b] \rightarrow \mathbb{K}$  be a Lipschitzian mapping with the constant  $L > 0$ . Then*

$$(1.14) \quad |\Psi(f, u; a, b)| \leq \frac{1}{2^p} HL (b - a)^{p+1}.$$

**Corollary 5.** *Let  $f$  be as above and  $G : [a, b] \rightarrow \mathbb{R}$  be the cumulative distribution function of a certain random variable  $X$ . Then*

$$(1.15) \quad \left| \frac{f(a) + f(b)}{2} - \int_a^b f(t) dG(t) \right| \leq \frac{1}{2^p} H (b - a)^p.$$

**Remark 1.** If we assume that  $g : [a, b] \rightarrow \mathbb{K}$  is continuous, then  $u(x) = \int_a^x g(t) dt$  is differentiable,  $u(b) = \int_a^b g(t) dt$ ,  $u(a) = 0$ , and  $\bigvee_a^b(u) = \int_a^b |g(t)| dt$ . Consequently, by (1.11), we obtain

$$(1.16) \quad \left| \frac{f(a) + f(b)}{2} \cdot \int_a^b g(t) dt - \int_a^b f(t) g(t) dt \right| \leq \frac{1}{2^p} H(b-a)^p \int_a^b |g(t)| dt.$$

The following theorem which complements, in a sense, the previous result also holds [13].

**Theorem 4.** Let  $f : [a, b] \rightarrow \mathbb{K}$  be a mapping of bounded variation on  $[a, b]$  and  $u : [a, b] \rightarrow \mathbb{K}$  be a  $p-H$ -Hölder type mapping, that is, it satisfies the condition:

$$(1.17) \quad |u(x) - u(y)| \leq H|x - y|^p \text{ for all } x, y \in [a, b],$$

where  $H > 0$  and  $p \in (0, 1]$  are given. Then we have the inequality:

$$(1.18) \quad |\Psi(f, u; a, b)| \leq \frac{1}{2^p} H(b-a)^p \bigvee_a^b(f).$$

The constant  $C = 1$  on the right hand side of (1.18) cannot be replaced by a smaller constant.

The following corollary is a natural consequence of the above result.

**Corollary 6.** Let  $f : [a, b] \rightarrow \mathbb{K}$  be as in Theorem 4 and  $u$  be an  $L$ -Lipschitzian mapping on  $[a, b]$ , that is,

$$(1.19) \quad |u(t) - u(s)| \leq L|t - s| \text{ for all } t, s \in [a, b],$$

where  $L > 0$  is fixed. Then we have the inequality

$$(1.20) \quad |\Psi(f, u; a, b)| \leq \frac{L}{2} (b-a) \bigvee_a^b(f).$$

**Remark 2.** If  $f : [a, b] \rightarrow \mathbb{R}$  is monotonic and  $u$  is of  $p-H$ -Hölder type, then the inequality (1.18) becomes:

$$(1.21) \quad |\Psi(f, u; a, b)| \leq \frac{1}{2^p} H(b-a) |f(b) - f(a)|.$$

In addition, if  $u$  is  $L$ -Lipschitzian, then the inequality (1.20) can be replaced by

$$(1.22) \quad |\Psi(f, u; a, b)| \leq \frac{L}{2} (b-a) |f(b) - f(a)|.$$

**Remark 3.** If  $f$  is Lipschitzian with a constant  $K > 0$ , then it is obvious that  $f$  is of bounded variation on  $[a, b]$  and  $\bigvee_a^b(f) \leq K(b-a)$ . Consequently, the inequality (1.18) becomes

$$(1.23) \quad |\Psi(f, u; a, b)| \leq \frac{1}{2^p} HK(b-a)^{p+1},$$

and the inequality (1.20) becomes

$$(1.24) \quad |\Psi(f, u; a, b)| \leq \frac{LK}{2} (b-a)^2.$$

We now point out some results in estimating the integral of a product.

**Corollary 7.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a mapping of bounded variation on  $[a, b]$  and  $g$  be continuous on  $[a, b]$ . Put  $\|g\|_\infty := \sup_{t \in [a, b]} |g(t)|$ . Then we have the inequality:*

$$(1.25) \quad \left| \frac{f(a) + f(b)}{2} \int_a^b g(s) ds - \int_a^b f(t) g(t) dt \right| \leq \frac{\|g\|_\infty}{2} (b-a) \bigvee_a^b(f).$$

**Remark 4.** *Now, if in the above corollary we assume that  $f$  is monotonic, then (1.25) becomes*

$$(1.26) \quad \left| \frac{f(a) + f(b)}{2} \int_a^b g(s) ds - \int_a^b f(t) g(t) dt \right| \leq \frac{\|g\|_\infty |f(b) - f(a)| (b-a)}{2},$$

and if in Corollary 7 we assume that  $f$  is  $K$ -Lipschitzian, then the inequality (1.25) becomes

$$(1.27) \quad \left| \frac{f(a) + f(b)}{2} \int_a^b g(s) ds - \int_a^b f(t) g(t) dt \right| \leq \frac{\|g\|_\infty K (b-a)^2}{2}.$$

The following corollary is also a natural consequence of Theorem 4.

**Corollary 8.** *Let  $f$  and  $g$  be as in Corollary 7. Put*

$$\|g\|_p := \left( \int_a^b |g(s)|^p ds \right)^{\frac{1}{p}}; p > 1.$$

*Then we have the inequality*

$$(1.28) \quad \left| \frac{f(a) + f(b)}{2} \int_a^b g(s) ds - \int_a^b f(t) g(t) dt \right| \leq \frac{1}{2^{\frac{p-1}{p}}} \|g\|_p (b-a)^{\frac{p-1}{p}} \bigvee_a^b(f).$$

## 2. THE RESULTS

The following theorem holds.

**Theorem 5.** *Let  $u : [a, b] \rightarrow \mathbb{R}$  be of  $H$ - $r$ -Hölder type, i.e., we recall this*

$$(2.1) \quad |u(x) - u(y)| \leq H|x - y|^r, \text{ for any } x, y \in [a, b] \text{ and some } H > 0,$$

*where  $r \in (0, 1]$  is given, and  $f : [a, b] \rightarrow \mathbb{R}$  is of bounded variation.*

*Then we have the inequality:*

$$(2.2) \quad \left| \int_a^b f(t) du(t) - [(u(b) - u(x))f(b) + (u(x) - u(a))f(a)] \right| \leq H \left[ \frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^r \bigvee_a^b(f) \leq H(b-a)^r \bigvee_a^b(f)$$

*for any  $x \in [a, b]$ .*

*The constant  $\frac{1}{2}$  is sharp in the sense that we cannot put a smaller constant instead.*

*Proof.* Using the integration by parts formula, we may state:

$$(2.3) \quad \begin{aligned} & \int_a^b (u(t) - u(x))df(t) \\ &= [u(b) - u(x)]f(b) - [u(a) - u(x)]f(a) - \int_a^b f(t)du(t). \end{aligned}$$

It is well known that if  $m : [a, b] \rightarrow \mathbb{R}$  is continuous and  $n : [a, b] \rightarrow \mathbb{R}$  is of bounded variation, the Riemann-Stieltjes integral  $\int_a^b m(t)dn(t)$  exists, and

$$\left| \int_a^b m(t)dn(t) \right| \leq \sup_{t \in [a, b]} |m(t)| \cdot \bigvee_a^b(n).$$

Thus,

$$\begin{aligned} & \left| \int_a^b (u(t) - u(x))df(t) \right| \\ & \leq \sup_{t \in [a, b]} |u(t) - u(x)| \bigvee_a^b(f) \leq \sup_{t \in [a, b]} \{H|t - x|^r\} \bigvee_a^b(f) \\ & = H \max\{|b - x|^r, |x - a|^r\} \bigvee_a^b(f) = H[\max(b - x, x - a)]^r \bigvee_a^b(f) \\ & = H \left[ \frac{1}{2}(b - a) + \left| x - \frac{a + b}{2} \right| \right]^r \bigvee_a^b(f). \end{aligned}$$

Finally, as

$$\left| x - \frac{a + b}{2} \right| \leq \frac{1}{2}(b - a) \text{ for any } x \in [a, b]$$

we get the last inequality in (2.2).

To prove the sharpness of the constant  $\frac{1}{2}$ , we assume that (2.2) holds with the constant  $c > 0$ , i.e.,

$$(2.4) \quad \begin{aligned} & \left| \int_a^b f(t)du(t) - [(u(b) - u(x))f(b) + (u(x) - u(a))f(a)] \right| \\ & \leq H \left[ c(b - a) + \left| x - \frac{a + b}{2} \right| \right]^r \bigvee_a^b(f). \end{aligned}$$

Choose  $u(t) = t$  which is of  $(1 - 1)$ -Hölder type and  $f : [a, b] \rightarrow \mathbb{R}$ ,  $f(t) = 0$  if  $t \in \{a, b\}$  and  $f(t) = 1$  if  $t \in (a, b)$ , which is of bounded variation, in (2.4).

We get:

$$|b - a| \leq 2 \left[ c(b - a) + \left| x - \frac{a + b}{2} \right| \right], \text{ for any } x \in [a, b].$$

For  $x = \frac{a + b}{2}$ , we get:

$$|b - a| \leq 2c(b - a), \text{ i.e. } c \geq \frac{1}{2}.$$

■

**Remark 5.** If  $u$  is Lipschitz continuous function, i.e.

$$|u(x) - u(y)| \leq L|x - y| \text{ for any } x, y \in [a, b], (\text{ and some } L > 0),$$

the inequality (2.2) becomes:

$$(2.5) \quad \left| \int_a^b f(t)du(t) - [(u(b) - u(x))f(b) + (u(x) - u(a))f(a)] \right| \\ \leq L \cdot \left[ \frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \cdot \bigvee_a^b(f) \leq L(b-a) \bigvee_a^b(f).$$

**Corollary 9.** If  $f$  is of bounded variation on  $[a, b]$  and  $u$  is absolutely continuous with  $u' \in L_\infty[a, b]$  then instead of  $L$  in (2.5) we can put

$$\|u'\|_\infty = \operatorname{ess\,sup}_{t \in [a, b]} |u'(t)|.$$

**Corollary 10.** If  $g : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable on  $[a, b]$  and if we choose  $u(t) = \int_a^t g(s)ds$ , then

$$(2.6) \quad \left| \int_a^b f(t)g(t)dt - f(b) \int_x^b g(s)ds - f(a) \int_a^x g(s)ds \right| \\ \leq \|g\|_\infty \left[ \frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b(f) \leq \|g\|_\infty (b-a) \bigvee_a^b(f).$$

**Remark 6.** If in (2.6) we choose  $x = \frac{a+b}{2}$ , we get the best inequality in the class, i.e.,

$$(2.7) \quad \left| \int_a^b f(t)g(t)dt - f(b) \int_{\frac{a+b}{2}}^b g(s)ds - f(a) \int_a^{\frac{a+b}{2}} g(s)ds \right| \\ \leq \frac{1}{2} \|g\|_\infty (b-a) \bigvee_a^b(f).$$

### 3. APPROXIMATING RIEMANN-STIELTJES INTEGRAL

Let  $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$  a division of  $[a, b]$ . Denote  $h_i := x_{i+1} - x_i$ , and  $\nu(I_n) = \sup_{i=0, n-1} h_i$  then construct the sums

$$(3.1) \quad S(f, u, I_n, \xi) = \sum_{i=0}^{n-1} [u(x_{i+1}) - u(\xi_i)]f(x_{i+1}) + \sum_{i=0}^{n-1} [u(\xi_i) - u(x_i)]f(x_i),$$

where  $\xi_i \in [x_i, x_{i+1}]$ ,  $i = \overline{0, n-1}$  and  $\xi = (\xi_0, \xi_1, \dots, \xi_{n-1})$ .

We can state the following theorem concerning the approximation of Riemann-Stieltjes integral:

**Theorem 6.** Let  $f, u$  be as in Theorem 5 and  $I_n, \xi$  as defined above. Then:

$$(3.2) \quad \int_a^b f(t)du(t) = S(f, u, I_n, \xi) + R(f, u, I_n, \xi)$$



when  $S(f, u, I_n, \xi)$  is defined by (3.1) and the remainder  $R(f, u, I_n, \xi)$  satisfies the estimate:

$$\begin{aligned}
 (3.3) \quad |R(f, u, I_n, \xi)| &\leq H \cdot \left[ \frac{1}{2} \nu(I_n) + \sup_{i=0, n-1} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right]^r \bigvee_a^b(f) \\
 &\leq H \cdot \nu^r(I_n) \bigvee_a^b(f).
 \end{aligned}$$

*Proof.* We apply (2.2) on  $[x_i, x_{i+1}]$  to get:

$$\begin{aligned}
 &\left| \int_{x_i}^{x_{i+1}} f(t) du(t) - [u(x_{i+1}) - u(\xi_i)]f(x_{i+1}) - [u(\xi_i) - u(x_i)]f(x_i) \right| \\
 &\leq H \cdot \left[ \frac{1}{2} h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right]^r \bigvee_{x_i}^{x_{i+1}}(f) \leq H \cdot h_i^r \bigvee_{x_i}^{x_{i+1}}(f).
 \end{aligned}$$

Summing on  $i$  from 0 to  $n-1$ , and using the generalised triangle inequality we get:

$$\begin{aligned}
 &\left| \int_a^b f(t) du(t) - S(f, u, I_n, \xi) \right| \\
 &\leq H \cdot \sum_{i=0}^{n-1} \left[ \frac{1}{2} h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right]^r \cdot \bigvee_{x_i}^{x_{i+1}}(f) \\
 &\leq H \sup_{i=0, n-1} \left[ \frac{1}{2} h_i + \left| \xi - \frac{x_i + x_{i+1}}{2} \right| \right]^r \bigvee_a^b(f) \\
 &\leq H \left[ \frac{1}{2} \nu(I_n) + \sup_{i=0, n-1} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right]^r \bigvee_a^b(f) \\
 &\leq H \nu^r(I_n) \bigvee_a^b(f),
 \end{aligned}$$

and the theorem is proved. ■

**Remark 7.** It is obvious that if  $\nu(I_n) \rightarrow 0$  then (3.2) provides an approximation for the Riemann-Stieltjes integral  $\int_a^b f(t) du(t)$ .

**Corollary 11.** If we consider the sum

$$\begin{aligned}
 &S_M(f, u, I_n) \\
 &= \sum_{i=0}^{n-1} \left[ u(x_{i+1}) - u\left(\frac{x_i + x_{i+1}}{2}\right) \right] f(x_{i+1}) + \sum_{i=0}^{n-1} \left[ u\left(\frac{x_i + x_{i+1}}{2}\right) - u(x_i) \right] f(x_i)
 \end{aligned}$$

then:

$$(3.4) \quad \int_a^b f(t) du(t) = S_M(f, u, I_n) + R_M(f, u, I_n)$$

and the remainder  $R_M(f, u, I_n)$  satisfies the estimate

$$(3.5) \quad |R_M(f, u, I_n)| \leq \frac{1}{2^r} H \nu^r(I_n) \bigvee_a^b(f).$$

The following corollary in approximating the integral  $\int_a^b f(t)g(t)dt$  holds.

**Corollary 12.** *If  $f, g$  are as in Corollary 10, then*

$$\int_a^b f(t)g(t)dt = P(f, g, I_n, \xi) + R_P(f, g, I_n, \xi)$$

where

$$P(f, g, I_n, \xi) = \sum_{i=0}^{n-1} f(x_{i+1}) \int_{\xi_i}^{x_{i+1}} g(s)ds + \sum_{i=0}^{n-1} f(x_i) \int_{x_i}^{\xi_i} g(s)ds.$$

and the remainder  $R_P(f, g, I_n, \xi)$  satisfies the estimate:

$$\begin{aligned} |R_P(f, g, I_n, \xi)| &\leq \|g\|_\infty \left[ \frac{1}{2} \nu(I_n) + \sup_{i=0, n-1} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \bigvee_a^b(f) \\ &\leq \|g\|_\infty \nu(I_n) \bigvee_a^b(f). \end{aligned}$$

**Remark 8.** *If in the above corollary we choose  $\xi_i = \frac{x_i + x_{i+1}}{2}$  ( $i = 0, n-1$ ) then we get the best formula in the class, i.e.,*

$$P_M(f, g, I_n, \xi) = \sum_{i=0}^{n-1} f(x_{i+1}) \int_{\frac{x_i + x_{i+1}}{2}}^{x_{i+1}} g(s)ds + \sum_{i=0}^{n-1} f(x_i) \int_{x_i}^{\frac{x_i + x_{i+1}}{2}} g(s)ds$$

and

$$R_{P_M}(f, g, I_n, \xi) \leq \frac{1}{2} \|g\|_\infty \nu(I_n) \bigvee_a^b(f).$$

#### 4. APPLICATION FOR SPECIAL MEANS

Consider the means:

1. Arithmetic mean

$$A(a, b) := \frac{a+b}{2}; a, b \geq 0;$$

2. Geometric mean

$$G(a, b) := \sqrt{ab}; a, b \geq 0;$$

3. Harmonic mean

$$H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}; a, b > 0;$$

4. Logarithmic mean

$$L(a, b) := \begin{cases} \frac{b-a}{\ln b - \ln a}; & a, b > 0, a \neq b \\ a, & a = b. \end{cases}$$

## 5. Identric mean

$$I(a, b) := \begin{cases} \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{\frac{1}{b-a}}; & a, b > 0, a \neq b \\ a, & a = b. \end{cases}$$

6.  $p$ -Logarithmic mean

$$L_p(a, b) := \begin{cases} \left[ \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}}; & a, b > 0, a \neq b \\ a, & a = b. \end{cases}, \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$

It is well known that  $L_p(a, b)$  is monotonically increasing as a function of  $p \mapsto L_p(a, b)$  denoting that  $L_{-1} = L$  and  $L_0 = I$ .

In Section 2 we proved the following inequality:

$$\begin{aligned} & \left| \int_a^b f(t)g(t)dt - f(b) \int_x^b g(s)ds - f(a) \int_a^x g(s)ds \right| \\ & \leq \|g\|_\infty \left[ \frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b(f) \leq \|g\|_\infty (b-a) \bigvee_a^b(f). \end{aligned}$$

We can use this inequality in the sequel for different selections of  $f$  and  $g$ .

1. If we choose:  $f(x) = x^p$  and  $g(x) = x^q, x \in [a, b], a, b > 0$  we get the inequalities:

$$\begin{aligned} & |(b-a)L_{p+q}^{p+q}(a, b) - b^p(b-x)L_q^q(x, b) - a^p(x-a)L_q^q(a, x)| \\ & \leq b^q p(b-a)^2 L_{p-1}^{p-1}(a, b) \end{aligned}$$

for any  $q > 0$  and

$$\begin{aligned} & |(b-a)L_{p+q}^{p+q}(a, b) - b^p(b-x)L_q^q(x, b) - a^p(x-a)L_q^q(a, x)| \\ & \leq a^q p(b-a)^2 L_{p-1}^{p-1}(a, b) \end{aligned}$$

for any  $q < 0, q \neq -1$ . Particularly, for  $x = A(a, b)$  we obtain:

$$\begin{aligned} & |2L_{p+q}^{p+q}(a, b) - b^p L_q^q(A(a, b), b) - a^p L_q^q(a, A(a, b))| \\ & \leq b^q p(b-a) L_{p-1}^{p-1}(a, b) \end{aligned}$$

for any  $q > 0$ , respectively,

$$\begin{aligned} & |2L_{p+q}^{p+q}(a, b) - b^p L_q^q(A(a, b), b) - a^p L_q^q(a, A(a, b))| \\ & \leq a^q p(b-a) L_{p-1}^{p-1}(a, b) \end{aligned}$$

for any  $q < 0, q \neq -1$ .

2. If we choose:  $f(x) = x^p$  and  $g(x) = \frac{1}{x}, x \in [a, b], a, b > 0$  we get the inequality:

$$\begin{aligned} & |(b-a)L_{p-1}^{p-1}(a, b) - b^p(b-x)L_{-1}^{-1}(x, b) - a^p(x-a)L_{-1}^{-1}(a, x)| \\ & \leq \frac{p}{a}(b-a)^2 L_{p-1}^{p-1}(a, b). \end{aligned}$$

Particularly, for  $x = A(a, b)$  we obtain:

$$\left| 2L_{p-1}^{p-1}(a, b) - b^p L_{-1}^{-1}(A(a, b), b) - a^p L_{-1}^{-1}(a, A(a, b)) \right| \leq \frac{p}{a} (b-a) L_{p-1}^{p-1}(a, b).$$

3. If we choose:  $f(x) = x^p$  and  $g(x) = \ln x, x \in [a, b], a, b > 0$  we get the inequality:

$$\begin{aligned} & \left| \frac{b-a}{p+1} [(p \ln b + \ln b - 1) L_p^p(a, b) + a^{p+1} L_{-1}^{-1}(a, b)] \right. \\ & \quad \left. - b^p (b-x) \ln(L_0(x, b)) - a^p (x-a) \ln(L_0(a, x)) \right| \\ & \leq p(b-a)^2 (\ln b) L_{p-1}^{p-1}(a, b). \end{aligned}$$

Particularly, for  $x = A(a, b)$  we obtain:

$$\begin{aligned} & \left| \frac{2}{p+1} [(p \ln b + \ln b - 1) L_p^p(a, b) + a^{p+1} L_{-1}^{-1}(a, b)] \right. \\ & \quad \left. - b^p \ln(L_0(A(a, b), b)) - a^p \ln(L_0(a, A(a, b))) \right| \\ & \leq p(b-a) \ln b L_{p-1}^{p-1}(a, b). \end{aligned}$$

4. If we choose:  $f(x) = \frac{1}{x}$  and  $g(x) = x^q, x \in [a, b], a, b > 0$  we get the inequalities:

$$\left| G^2(a, b) (b-a) L_{q-1}^{q-1}(a, b) - a(b-x) L_q^q(x, b) - b(x-a) L_q^q(a, x) \right| \leq (b-a)^2 b^q$$

for any  $q > 0$  and

$$\left| G^2(a, b) (b-a) L_{q-1}^{q-1}(a, b) - a(b-x) L_q^q(x, b) - b(x-a) L_q^q(a, x) \right| \leq (b-a)^2 a^q$$

for any  $q < 0, q \neq -1$ .

Particularly, for  $x = A(a, b)$  we obtain:

$$\left| 2G^2(a, b) L_{q-1}^{q-1}(a, b) - a L_q^q(A(a, b), b) - b L_q^q(a, A(a, b)) \right| \leq (b-a) b^q$$

for any  $q > 0$ , respectively:

$$\left| 2G^2(a, b) L_{q-1}^{q-1}(a, b) - a L_q^q(A(a, b), b) - b L_q^q(a, A(a, b)) \right| \leq (b-a) a^q$$

for any  $q < 0, q \neq -1$ .

5. If we choose:  $f(x) = \frac{1}{x}$  and  $g(x) = \frac{1}{x}$  we get the inequality:

$$\left| b-a - a(b-x) L_{-1}^{-1}(x, b) - b(x-a) L_{-1}^{-1}(a, x) \right| \leq \frac{(b-a)^2}{a}.$$

Particularly, for  $x = A(a, b)$  we obtain:

$$\left| 2 - a L_{-1}^{-1}(A(a, b), b) - b L_{-1}^{-1}(a, A(a, b)) \right| \leq \frac{b-a}{a}.$$

6. If we choose:  $f(x) = \frac{1}{x}$  and  $g(x) = \ln x$  we get the inequality:

$$\begin{aligned} & \left| G^2(a, b) \cdot \frac{b-a}{2} \cdot \ln(G^2(a, b)) \cdot L_{-1}^{-1}(a, b) - a(b-x) \ln(L_0(x, b)) \right. \\ & \quad \left. - b(x-a) \ln(L_0(a, x)) \right| \\ & \leq (b-a)^2 \ln b. \end{aligned}$$

Particularly, for  $x = A(a, b)$  we obtain:

$$|G^2(a, b) \ln(G^2(a, b)) L_{-1}^{-1}(a, b) - a \ln(L_0(a, A(a, b)))| \leq (b-a) \ln b$$

7. If we choose:  $f(x) = \ln x$  and  $g(x) = x^q$  we get the inequalities:

$$\begin{aligned} & \left| \frac{b-a}{q+1} [(q \ln b + \ln b - 1) L_q^q(a, b) + a^{q+1} L_{-1}^{-1}(a, b)] \right. \\ & \quad \left. - (\ln b)(b-x) L_q^q(x, b) - (\ln a)(a-x) L_q^q(a, x) \right| \\ & \leq (b-a)^2 b^q L_{-1}^{-1}(a, b) \text{ for any } q > 0, \end{aligned}$$

and

$$\begin{aligned} & \left| \frac{b-a}{q+1} [(q \ln b + \ln b - 1) L_q^q(a, b) + a^{q+1} L_{-1}^{-1}(a, b)] \right. \\ & \quad \left. - (\ln b)(b-x) L_q^q(x, b) - (\ln a)(a-x) L_q^q(a, x) \right| \\ & \leq (b-a)^2 a^q L_{-1}^{-1}(a, b) \text{ for any } q < 0, q \neq -1. \end{aligned}$$

Particularly, for  $x = A(a, b)$  we obtain:

$$\begin{aligned} & \left| \frac{2}{q+1} [(q \ln b + \ln b - 1) L_q^q(a, b) + a^{q+1} L_{-1}^{-1}(a, b)] \right. \\ & \quad \left. - \ln b L_q^q(A(a, b), b) - \ln a L_q^q(a, A(a, b)) \right| \\ & \leq (b-a) b^q L_{-1}^{-1}(a, b) \text{ for any } q > 0, \end{aligned}$$

respectively:

$$\begin{aligned} & \left| \frac{2}{q+1} [(q \ln b + \ln b - 1) L_q^q(a, b) + a^{q+1} L_{-1}^{-1}(a, b)] \right. \\ & \quad \left. - \ln b L_q^q(A(a, b), b) - \ln a L_q^q(a, A(a, b)) \right| \\ & \leq (b-a) a^q L_{-1}^{-1}(a, b) \text{ for any } q < 0, q \neq -1. \end{aligned}$$

8. If we choose:  $f(x) = \ln x$  and  $g(x) = \frac{1}{x}$  we get the inequality:

$$\begin{aligned} & \left| \frac{b-a}{2} \ln G^2(a, b) L_{-1}^{-1}(a, b) - (b-x) \ln b L_{-1}^{-1}(x, b) - (a-x) \ln a L_{-1}^{-1}(a, x) \right| \\ & \leq \frac{(b-a)^2}{a} L_{-1}^{-1}(a, b). \end{aligned}$$

Particularly, for  $x = A(a, b)$  we obtain:

$$\begin{aligned} & \left| \ln G^2(a, b) L_{-1}^{-1}(a, b) - \ln b L_{-1}^{-1}(A(a, b), b) - \ln a L_{-1}^{-1}(a, A(a, b)) \right| \\ & \leq \frac{b-a}{a} L_{-1}^{-1}(a, b). \end{aligned}$$

9. If we choose:  $f(x) = \ln x$  and  $g(x) = \ln x$  we get the inequality:

$$\begin{aligned} & \left| \frac{b-a}{G^2(a, b)} [b(\ln a^a b^b - 2) \ln(L_0(a, b)) + b \ln a^a b^b - \ln^2 b^b] \right. \\ & \quad \left. - (b-x) \ln b \ln(L_0(x, b)) - (x-a) \ln a \ln(L_0(a, x)) \right| \\ & \leq (b-a)^2 \ln b L_{-1}^{-1}(a, b). \end{aligned}$$

Particularly, for  $x = A(a, b)$  we obtain:

$$\begin{aligned} & \left| \frac{2}{G^2(a, b)} [b(\ln a^a b^b - 2) - \ln(L_0(a, b)) + b \ln a^a b^b - (\ln b^b)^2] \right. \\ & \quad \left. - \ln a \ln(L_0(a, A(a, b))) \right| \\ & \leq (b-a) \ln b L_{-1}^{-1}(a, b). \end{aligned}$$

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